Probability Theory

1. Prove: (a) $\bigcap_{n=1}^{\infty}(\frac{1}{3}-\frac{1}{3n},\frac{1}{3}+\frac{1}{3n}) = \{\frac{1}{3}\}$ (6%)

(b) Using (a) to explain that $P\{\frac{1}{3}\}=0$ (i. e. Suppose that we select a random point from the interval (0, 1), try to explain the probability of selecting the point $\frac{1}{3}$ is zero. (Hint: $\lim_{n\to\infty} P(E_n) = P(\lim_{n\to\infty} E_n)$)(10%)

2.(a) Write down the formula of Law of Total Probability(6%)

(b) Using (a) to explain: suppose that seven coins, of which exactly three are gold, are distributed among seven persons, one each, at random, and one by one. Are the chances of getting a gold coin equal for all participants? (10%)

3. A certain basketball player makes a foul shot with probability 0.61. Determine for what value of k the possibility of k baskets in 12 shots is maximum, and find this maximum probability. (10%)

4. Mr. Chen owns two appliance stores. In store 1 the number of TV sets sold by a salesperson is, on average, 9 per week with a standard deviation of five. In store 2 the number of TV sets sold by a salesperson is, on average, 8 with a standard deviation of four. Mr. Chen has a position open for a person to sell TV sets. There are two applicants. Mr. Chen asked one of them to work in store 1 and the other in store 2, each for one week. Both of the salesperson in store 1 and 2 sold 7 sets. Based on this information, which person should Mr. Chen hire? Please explain: (10%)

5. Let the joint probability density function of random variables *X* and *Y* be given by

$$f(x,y)=\begin{cases}\frac{1}{2}ye^{-x}, \text{if } x>0, 0<y<2.\\ 0, \text{otherwise.}\end{cases}$$

Find the marginal probability density functions of *X* and *Y*. (12%)

6. A man invites his fiancee to an elegant hotel for a Sunday brunch. They decide to meet in the lobby of the hotel between 11:20 A.M. and 12 noon. If they arrive at random times during this period, what is the probability that the first to arrive has to wait at least 10 minutes? (10%)

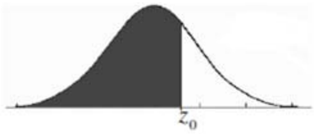
7. Customers arrive at a post office at a Poisson rate of four per minute. What is the probability that the next customer does not arrive during the next 1 minutes? (10%)

8. (a) Give the statement of the <u>Central Limit Theorem.</u> (6%)

(b) Using (a) to solve: If 20 random numbers are selected independently from the interval (0,1), what is the approximate probability that the sum of these numbers is at least eight? (10%) (Hint: a. $\sqrt{20}\approx 4.472$, $\sqrt{12}\approx 3.464$, b. check the table on the next page)

Table 2 Area under the Standard Normal Distribution to the Left of z_0: Positive z_0

$$\Phi(z_0) = P(Z \leq z_0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_0} e^{-x^2/2}\,dx$$



z_0	0	1	2	3	4	5	6	7	8	9
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7703	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9889	.9889	.9889	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998
3.6	.9998	.9998	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.7	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999
3.8	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999	.9999

Discrete Mathematics

CSE PhD Qualifying Exam

July 2022

1. (20%) Let $G = (V, E)$ be a connected graph and $|V| = n$. What are the minimum values of $|E|$ so that G can be constructed, respectively, as
   (a) a complete bipartite graph;
   (b) a cycle of length 8;
   (c) a spanning tree; and
   (d) a regular graph?

2. (10%) Use **strong induction** to show every positive integer n can be written as sum of distinct powers of two, that is, as a sum of a subset of integers $2^0 = 1, 2^1 = 2, 2^2 = 4$ and so on.
   Hint: for the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.

3. (10%) Let $A = \{1, 2, 3, 4, 5\} \times \{1, 2, 3, 4, 5\}$, and define a binary relation $\mathcal{R}$ on A as follows:
   $$(x_1, y_1)\mathcal{R}(x_2, y_2) \text{ if and only if } x_1 + y_1 = x_2 + y_2.$$
   (a) Verify that $\mathcal{R}$ is an equivalence relation.
   (b) Find the equivalence class that includes.

4. (10%) A positive rational number can be expressed as p/q, where p and q are two positive integers with $gcd(p, q) = 1$. Prove that $3^{1/2}$ is not a rational number.