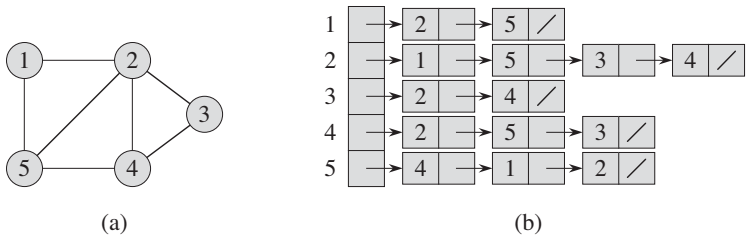
Algorithms
CSE PhD Qualifying Exam
January 2023

1. (20%) Illustrate the operation of the following sorting algorithms respectively on the array $A = \langle 8, 4, 7, 4, 0, 3, 9, 6, 9, 3, 5 \rangle$, where $A[j] \in \{0, 1, \ldots, 9\}$ for $1 \leq j \leq 11$. Which of them are stable sorting algorithms? Which of them are in-place sorting algorithms?

   (a) MERGE SORT

   (b) QUICKSORT

2. (20%) Illustrate the progresses of BFS and DFS, respectively, starting from vertex 4 on the following graph.



(a) (b)

3. (20%) Suppose you are given two sets A and B, each containing n positive integers. You can choose to reorder each set however you like. After reordering, let a_i be the ith element of set A, and let b_i be the ith element of set B. You then receive a payoff of $\prod_{i=1}^{n} a_i^{b_i}$.

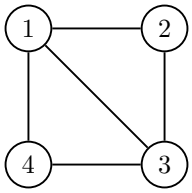
   (a) Give an algorithm that will maximize your payoff.

   (b) Prove that your algorithm maximizes the payoff, and state its running time.

4. (20%) Give an algorithm for determining if a graph is two-colorable, i.e. if it is possible to color every vertex red or blue so that no two vertices of the same color have an edge between them. Your algorithm should run in time $O(n + m)$, where n is the number of vertices and m is the number of edges in the graph. You should assume that the graph is undirected and that the input is presented in adjacency-list form.

5. (20%) Polynomial-Time Reductions:
   In the DOMINATING SET problem, we are given an undirected graph $G = (V, E)$ with n vertices and a number k ($1 \leq k \leq n$). A vertex u dominates itself and all of its neighbors. That is, vertex u dominates vertex v if $v = u$ or v is adjacent to u. A set S of the vertices is called a *dominating set* if every vertex $v \in V$ is dominated by at least one vertex $u \in S$. DOMINATING SET problem asks you to check whether there is a dominating set of size k in graph G. It is well-known that DOMINATING SET is an NP-complete problem.

   In this problem, we consider a variant called DOUBLE DOMINATING SET. The input is an undirected graph $G' = (V', E')$ with n' vertices, and a number k' ($1 \leq k' \leq n'$). A set $S' \subset V'$ is called a *double dominating set*, if every vertex $v \in V'$ is dominated by at least two vertices in S'.



Example: For the graph on the left, vertex $\{1\}$ is a dominating set of size 1; vertices $\{2, 4\}$ form a dominating set of size 2. However, neither $\{1\}$ nor $\{2, 4\}$ is a double dominating set. The set $\{1, 3\}$ is a double dominating set of size 2.

(a) To show DOUBLE DOMINATING SET is NP-hard based on the fact that DOMINATING SET is NP-complete, what is the correct direction of reduction?
(Please answer in the form A to B)

(b) Prove that DOUBLE DOMINATING SET is in NP.

(c) Do a reduction (related to the DOMINATING SET problem) to show DOUBLE DOMINATING SET is NP-hard.
Hint: The intended solution only creates 2 extra vertices in the new instance.